Phys 410
Spring 2013
Lecture #21 Summary
11 March, 2013

Constrained systems are common in physics, and their dynamics can be advantageously solved by the Lagrangian method. Examples include the pendulum, the Atwood machine, and a bead on a wire. We considered the pendulum problem in detail. The constraint is that the length of the pendulum ℓ is fixed, so that the x- and y-coordinates of the bob are not independent, but constrained so that $\ell = \sqrt{x^2 + y^2}$. We can incorporate this constraint by adopting a new independent variable to describe the position of the bob, namely the angle that the pendulum makes with the vertical, ϕ. In terms of this generalized coordinate, the Lagrangian becomes $\mathcal{L}(\phi, \dot{\phi}) = \frac{m}{2}\ell^2\dot{\phi}^2 - mg\ell(1 - \cos\phi)$. Lagrange's equation gives $-mg\ell\sin\phi = m\ell^2\ddot{\phi}$, which relates the torque due to gravity on the bob to the time rate of change of the angular momentum of the bob, or the moment of inertia ($m\ell^2$) times the angular acceleration ($\ddot{\phi}$). Note that the force of constraint (namely the tension in the rod supporting the bob) never played a role in the analysis. Once the appropriate generalized coordinate is identified, the associated constraining force disappears from the discussion!

Generalized coordinates and constrained systems are important for Lagrangian dynamics. Consider a system consisting of N particles, with positions $\overrightarrow{r_\alpha}$, with $\alpha = 1, \dots N$. The parameters $q_1, q_2, \dots q_n$ are a set of generalized coordinates if each position $\overrightarrow{r_\alpha}$ can be expressed as a function of $q_1, q_2, \dots q_n$, and possibly time t as, $\overrightarrow{r_\alpha} = \overrightarrow{r_\alpha}(q_1, q_2, \dots q_n, t)$for $\alpha = 1, \dots N$, and the inverse $q_i = q_i(\overrightarrow{r_1}, \overrightarrow{r_2}, \dots \overrightarrow{r_N}, t)$ for $i = 1, 2, \dots n$ can also be written. For particles in three dimensions, $n \leq 3N$. If $n < 3N$, then the system is said to be constrained. The number of degrees of freedom of a system is the number of coordinates that can be independently varied in a small displacement. The simple pendulum is constrained and has one degree of freedom. The double pendulum is constrained and has two degrees of freedom. One can show (the proof is in Taylor) that constrained systems obey the Lagrange equations when their Lagrangian is written in terms of the generalized coordinates of the system.

We concluded by doing the example of the Atwood machine for a frictionless and inertia-less pulley supporting two different masses. The masses can each move in one dimension (which we called x and y), and their motion is constrained because they are on either end of a string of fixed length. The constraint is that the string length is $\ell = x + y + \pi R$, where R is the radius of the pulley. With this constraint incorporated, the Lagrangian can be written as $\mathcal{L}(x, \dot{x}) = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx + const$. Note that the constant plays no role in the dynamics since it disappears when both of the derivatives ($\frac{\partial\mathcal{L}}{\partial x}, \frac{\partial\mathcal{L}}{\partial\dot{x}}$) are taken. The resulting equation of motion is $\ddot{x} = g\frac{m_1 - m_2}{m_1 + m_2}$. Again note that the constraining force (the tension in the string) was never mentioned or considered in the process. The tension is essential to the traditional Newton's second law approach to solving this problem.